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'Surface' Element on the Light Cone

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Abstract

We find an explicit form for the vector 'surface' element on the light cone in Minkowski space by a limiting process from that of a space-like pseudosphere. We use it to discuss more precisely the confinement of the radiation emitted by a charged particle between two light cones.

1. *Introduction*

A three-dimensional 'surface' element in Minkowski space is a four-vector $d\sigma_\mu$, which usually can be written in the form

$$d\sigma_\mu = n_\mu d\sigma \tag{1.1}$$

where n_μ is the unit normal to the surface. However, on the light cone, or at points on other surfaces where the normal is a null vector, such a unit vector does not exist (the components tend to infinity) and $d\sigma$ vanishes (Synge, 1956; Rohrlich, 1965), but $d\sigma_\mu$ is finite.

In order to give a precise definition to the energy and momentum of the radiation emitted by an accelerated point charge for a segment of its world line (Schild, 1960), it has been found necessary to show that

$$\lim_{\rho \to \infty} \int_{\sigma_L} \Theta_{\mu\nu} d\sigma_\mu = 0 \tag{1.2}$$

where $\Theta_{\mu\nu}$ is the symmetrized stress-energy tensor for the electromagnetic field, σ_L is a band on a light cone, and the limit $\rho \to \infty$ indicates that the spatial dimensions tend to infinity. To discuss this integral in more detail, it is useful to have an explicit expression for $d\sigma_\mu$ that we derive in Section 2 by a limiting process from $d\sigma_\mu$ for a space-like pseudosphere. In Section 3 we show that the integral actually vanishes.

We use the time-favoring metric $g_{\mu\nu}$ whose non-zero components are

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \tag{1.3}$$

the modified summation convention for repeated Greek sub-indices

$$a\,.\,b = a_\mu b_\mu = a_0 b_0 - \mathbf{a}\,.\,\mathbf{b} = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \tag{1.4}$$

and we set the speed of light $c = 1$.

2. *Surface Element on the Light Cone*

We first determine the surface element on the space-like pseudosphere with origin at ξ and radius λ

$$(x - \xi)^2 = \lambda^2 \tag{2.1}$$

and then we let the (real) constant λ go to zero. The unit normal at a point x is given by

$$n_\mu = (x_\mu - \xi_\mu)/\lambda \tag{2.2}$$

and the vector surface element is given by

$$d\sigma_\mu = n_\mu\, d\sigma \tag{2.3}$$

In order to determine $d\sigma$, we introduce curvilinear coordinates on the pseudosphere, and we are careful to choose them in such a way that they will still be useful when $\lambda \to 0$. We take

$$x_0 = \xi_0 + \eta \tag{2.4}$$

$$x_1 = \xi_1 + (\eta^2 - \lambda^2)^{1/2} \sin\theta \cos\phi \tag{2.5}$$

$$x_2 = \xi_2 + (\eta^2 - \lambda^2)^{1/2} \sin\theta \sin\phi \tag{2.6}$$

$$x_3 = \xi_3 + (\eta^2 - \lambda^2)^{1/2} \cos\theta \tag{2.7}$$

and, calculating the metric element induced in this space by the Minkowski metric, we find

$$\begin{aligned} ds^2 &= dx_\mu\, dx_\mu \\ &= [\lambda^2/(\eta^2 - \lambda^2)]\, d\eta^2 - (\eta^2 - \lambda^2)(d\theta^2 + \sin^2\theta\, d\phi^2) \end{aligned} \tag{2.8}$$

hence the surface element is

$$d\sigma = \lambda(\eta^2 - \lambda^2)^{1/2} \sin\theta\, d\mu\, d\theta\, d\phi \tag{2.9}$$

When $\lambda \to 0$, the metric becomes singular, as expected, and $d\sigma \to 0$. Nevertheless, the vector surface element

$$d\sigma_\mu = (x_\mu - \xi_\mu)(\eta^2 - \lambda^2)^{1/2} \sin\theta\, d\eta\, d\theta\, d\phi \tag{2.10}$$

remains finite in this limit, and can be used in the calculations of the next section. We can also compute the surface element directly from

$$d\sigma_\mu = \epsilon_{\mu\nu\lambda\rho} \frac{\partial x^\nu}{\partial \eta} \frac{\partial x^\lambda}{\partial \theta} \frac{\partial x^\rho}{\partial \phi}\, d\eta\, d\theta\, d\phi \tag{2.11}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric numerical tensor (its components are 0, ±1) and the x_ν are given by equations (2.4) through (2.7) with λ set equal to zero. Both methods give the same result for the metric (1.3), but they are not equivalent in a general case.

3. *Integral Over the Light Cone*

We consider the radiation emitted by a point source of prescribed trajectory in Minkowski space given by the parametric equations

$$\xi_\mu = \xi_\mu(\tau) \tag{3.1}$$

where we choose the parameter τ to be the proper time, so that the velocity $u_\mu = d\xi_\mu/d\tau$ satisfies $u^2 = 1$. The stress-energy tensor is (Schild, 1960)

$$\begin{aligned}\Theta_{\mu\nu}(x) = -\frac{e^2}{16\pi^2\epsilon_0}\Bigg\{&\frac{[(1-R.w)^2 + w^2(R.u)^2]R_\mu R_\nu}{(R.u)^6}\\ &-\frac{(1-R.w)(u_\mu R_\nu + u_\nu R_\mu)}{(R.u)^5} - \frac{w_\mu R_\nu + w_\nu R_\mu}{(R.u)^4}\Bigg\}\\ &-\frac{e^2}{32\pi^2\epsilon_0}\frac{g_{\mu\nu}}{(R.u)^4}\end{aligned} \tag{3.2}$$

where w_μ is the acceleration $du_\mu/d\tau$

$$R_\mu = x_\mu - \xi_\mu \tag{3.3}$$

and τ is determined as a function of x by

$$R^2 = 0, \qquad R_0 > 0 \tag{3.4}$$

that is, x is on the forward light cone with vertex at $\xi(\tau)$. Equation (2.10) gives the surface element on the light cone

$$d\sigma_\mu = \mathrm{R}_\mu \eta \sin\theta \, d\eta \, d\theta \, d\phi \tag{3.5}$$

Then, to show that the radiation remains confined between light cones, we have to prove the relation (1.2). We use the fact that R is a null vector to obtain

$$\int_{\sigma_L} \Theta_{\rho\nu}\, d\sigma_\mu = \frac{e^2}{32\pi^2\epsilon_0}\int_{\sigma_L}\frac{d\sigma_\nu}{(R.u)^4} \tag{3.6}$$

we see that only terms of the order of ρ^{-4} survive in $\Theta_{\mu\nu}$, where $\rho = R.u$. If the width of the band σ_L, or the range of integration over η, remains finite as $\rho \to \infty$, the integral goes to zero as ρ^{-2}; on the other hand, if this width increases as ρ does, the integral goes to zero as ρ^{-1} only, which is Schild's result.

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References

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Schild, A. (1960). *Journal of Mathematical Analysis and Applications,* **1**, 127. See also Rohrlich (1965), Sec. 5-1, pp. 106 ff. A more detailed discussion and helpful figures are found in these references.

Synge, J. L. (1956). *Relativity: The Special Theory*, Appendix D. North-Holland Publishing Company, Amsterdam.